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Behaviour of biaxial nematics in the presence of electric and magnetic fields
Evidence of bistability

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The behaviour of a biaxial nematic liquid crystal in the presence of electric and magnetic fields is discussed. In terms of the values of the magnetic susceptibilities and the dielectric permittivities, each biaxial nematic compound can be classified to belong to one of thirty-six different states. These states can be grouped together into three different classes, denoted by us as type A, B and C. The states belonging to each class exhibit a different qualitative behaviour in the presence of perpendicular electric and magnetic fields. While type A biaxial nematics always exhibit the same stable configuration in the presence of the fields, type B and C biaxial nematics exhibit two possible stable equilibrium configurations. Which of these is stable is determined by the magnitudes of the applied fields. The exchange of stability for type B systems can be modelled as a second order transition, while the exchange of stability for type C systems is of first order. In addition, the latter type can develop a bistable behaviour if certain conditions for the magnitudes of the electric and magnetic fields are fulfilled.

1. Introduction

Since the discovery of the biaxial nematic phase in a multicomponent system by Yu and Saupe [1] in 1980, a number of papers investigating the rheological properties of this system have appeared [2–7]. These papers together give a fairly complete description of the mathematical structure of the elastic as well as the hydrodynamic theory of a biaxial nematic system. The paper by Carlsson *et al.* [7] gives a derivation of the viscous stress tensor, showing the connection with the Leslie–Ericksen stress tensor [8, 9] of uniaxial nematics, and also provides a thorough investigation of the flow properties of biaxial nematics under the influence of this stress tensor.

Recently the biaxial nematic phase has been found [10–12] to also exist in thermotropic systems. This fact makes this phase more accessible to experimental investigations and in the future we would expect reports to appear regarding its elastic and viscous behaviour as well as attempts to measure the basic material parameters of the system. One important tool when performing this type of experiment is the possibility of controlling the system by means of electric and magnetic fields. However, it turns out that the behaviour of the biaxial nematic phase when subjected to electric

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and magnetic fields is more complex than might be expected. It has been suggested by Chandrasekhar [13] that biaxial nematic systems can be aligned by the use of two crossed electric and magnetic fields. In this paper we show that although the idea of Chandrasekhar is correct it has to be applied with some caution to be useful. With regard to the values of the magnetic susceptibilities and the dielectric permittivities, each biaxial nematic compound can be classified to belong to one of thirty-six different states. We demonstrate that these states can be grouped together into three classes, the states belonging to each class exhibiting a unique behaviour in the presence of crossed electric and magnetic fields, and introduce the concept type A, B, and C, respectively, to distinguish these different types of behaviour. Also we find that one of these types can develop bistability if certain conditions for the magnitudes of the electric and magnetic fields are fulfilled.

2. The electromagnetic free energy density and the electromagnetic torque

The symmetry of the biaxial nematic phase can be visualized as a plate with sides a, b and c as shown by figure 1, and without loss of generality we can assume $a > b > c$. In order to describe the orientation of the plate we introduce three orthogonal unit vectors $\hat{\mathbf{n}}$, $\hat{\mathbf{m}}$ and $\hat{\mathbf{l}}$ as shown. These three unit vectors are subject to the constraints indicated, and two of them are sufficient to specify the order of the system unambiguously. In our choice one of them essentially corresponds to the director of the uniaxial phase and is called the long director, $\hat{\mathbf{n}}$. The rotation of the biaxial plate around the long director is uniquely determined by $\hat{\mathbf{m}}$, denoted by us as the transverse director (cf figure 1).

In order to describe the dielectric and magnetic properties of the medium it is necessary to introduce three dielectric permittivities and threee magnetic susceptibilities, one for each principal axis of the biaxial plate. We denote these six constants ε_i

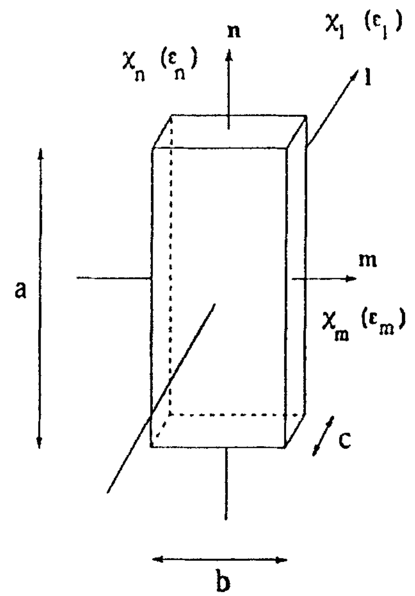

Figure 1. Introduction of the biaxial plate. In order to describe the system we introduce the long director $\hat{\mathbf{n}}$, the transverse director $\hat{\mathbf{m}}$ and the unit vector $\hat{\mathbf{l}} = \hat{\mathbf{n}} \times \hat{\mathbf{m}}$. For each axis of the biaxial plate there is a dielectric permittivity ε_i and a magnetic susceptibility χ_i.

and χ_i, respectively, where ε_i corresponds to the dielectric permittivity along the i axis and so on. Instead of discussing the problem in terms of these susceptibilities , it is frequently more convenient to introduce the corresponding dielectric and magnetic anisotropies,

$$\varepsilon_{ij}=\varepsilon_i-\varepsilon_j,\quad \chi_{ij}=\chi_i-\chi_j. \tag{1}$$

The application of a magnetic field **B** to the system leads to an induced magnetization **M** given by

$$\mathbf{M}=\mu_0^{-1}[\chi_n(\mathbf{B}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}+\chi_m(\mathbf{B}\cdot\hat{\mathbf{m}})\hat{\mathbf{m}}+\chi_l(\mathbf{B}\cdot\hat{\mathbf{l}})\hat{\mathbf{l}}], \tag{2}$$

where μ_0 is the permeability of free space. If instead the magnetic anisotropies introduced by equation (1) are used, we can write the induced magnetization as

$$\mathbf{M}=\mu_0^{-1}[\chi_{nl}(\mathbf{B}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}+\chi_{ml}(\mathbf{B}\cdot\hat{\mathbf{m}})\hat{\mathbf{m}}+\chi_l\mathbf{B}]. \tag{3}$$

The corresponding magnetic free energy density g_χ can be shown to be

$$g_\chi=-\int_0^{\mathbf{B}}\mathbf{M}\cdot d\mathbf{B}. \tag{4}$$

Introducing equation (2) or (3) into equation (4) we obtain

$$g_\chi=-\frac{1}{2\mu_0}[\chi_n(\mathbf{B}\cdot\hat{\mathbf{n}})^2+\chi_m(\mathbf{B}\cdot\hat{\mathbf{m}})^2+\chi_l(\mathbf{B}\cdot\hat{\mathbf{l}})^2] \tag{5}$$

$$=-\frac{1}{2\mu_0}[\chi_{nl}(\mathbf{B}\cdot\hat{\mathbf{n}})^2+\chi_{ml}(\mathbf{B}\cdot\hat{\mathbf{m}})^2+\chi_l B^2]. \tag{6}$$

In the same way we can derive the expression for the induced electric displacement

$$\mathbf{D}=\varepsilon_0[\varepsilon_n(\mathbf{E}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}+\varepsilon_m(\mathbf{E}\cdot\hat{\mathbf{m}})\hat{\mathbf{m}}+\varepsilon_l(\mathbf{E}\cdot\hat{\mathbf{l}})\hat{\mathbf{l}}] \tag{7}$$

$$=\varepsilon_0[\varepsilon_{nl}(\mathbf{E}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}+\varepsilon_{ml}(\mathbf{E}\cdot\hat{\mathbf{m}})\hat{\mathbf{m}}+\varepsilon_l\mathbf{E}], \tag{8}$$

and the electric free energy density

$$g_\varepsilon=-\frac{\varepsilon_0}{2}[\varepsilon_n(\mathbf{E}\cdot\hat{\mathbf{n}})^2+\varepsilon_m(\mathbf{E}\cdot\hat{\mathbf{m}})^2+\varepsilon_l(\mathbf{E}\cdot\hat{\mathbf{l}})^2], \tag{9}$$

$$=-\frac{\varepsilon_0}{2}[\varepsilon_{nl}(\mathbf{E}\cdot\hat{\mathbf{n}})^2+\varepsilon_{ml}(\mathbf{E}\cdot\hat{\mathbf{m}})^2+\varepsilon_l E^2]. \tag{10}$$

Most often the governing equations of the system are written down as a balance of torque equations. From electromagnetic theory it is well-known that the electric and magnetic torques acting on the system are given by

$$\mathbf{\Gamma}^\chi=\mathbf{M}\times\mathbf{B}, \tag{11}$$

$$\mathbf{\Gamma}^\varepsilon=\mathbf{D}\times\mathbf{E}. \tag{12}$$

3. Behaviour of a biaxial nematic in the presence of a single magnetic or electric field

Let us investigate the effect of applying a magnetic field to a biaxial nematic liquid crystal. As is seen from equations (5) and (9), for the case when an electric field is applied instead the results to be derived can be taken over directly by making the substitution

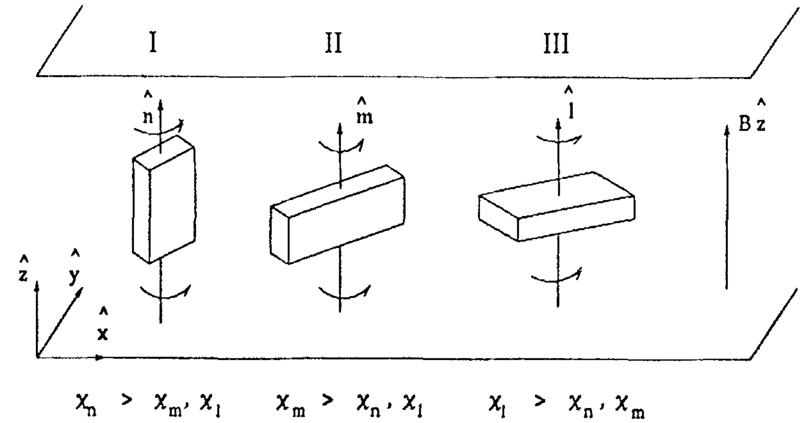

Figure 2. Ordering of a biaxial nematic in the presence of a magnetic field. The axis of the biaxial plate corresponding to the largest value of χ_i points in the direction of the field, while the other two axes are free to point in any compatible direction perpendicular to the field.

$B^2\chi_i/\mu_0 \to \varepsilon_0 E^2 \varepsilon_i$. Applying the magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ the induced magnetization (3) can be written as

$$\mathbf{M} = \frac{B}{\mu_0}[\chi_{nl} n_z \hat{\mathbf{n}} + \chi_{ml} m_z \hat{\mathbf{m}} + \chi_l \hat{\mathbf{z}}]. \tag{13}$$

The corresponding magnetic torque is calculated from equations (11) and (13) to be

$$\left.\begin{aligned} \Gamma_x^\chi &= \frac{B^2}{\mu_0}(\chi_{nl} n_y n_z + \chi_{ml} m_y m_z), \\ \Gamma_y^\chi &= -\frac{B^2}{\mu_0}(\chi_{nl} n_x n_z + \chi_{ml} m_x m_z) \\ \text{and} \qquad \Gamma_z^\chi &= 0. \end{aligned}\right\} \tag{14}$$

The equilibrium conditions for the system follow by demanding that the magnetic torque is zero;

$$\left.\begin{aligned} \chi_{nl} n_y n_z + \chi_{ml} m_y m_z &= 0 \\ \text{and} \qquad \chi_{nl} n_x n_z + \chi_{ml} m_x m_z &= 0. \end{aligned}\right\} \tag{15}$$

These equations have only three solutions (provided χ_n, χ_m and χ_l are all unequal) which are shown in figure 2. These solutions correspond to the situations for which each of the three principal axes of the biaxial plate points in the direction of the field. In order to determine which of these solutions is the stable one, it is necessary to compare the magnetic free energy density (see equation (6)) of the system in the three cases, this is given by

$$g_\chi = -\frac{B^2}{2\mu_0}[\chi_{nl} n_z^2 + \chi_{ml} m_z^2 + \chi_l]. \tag{16}$$

The three solutions of equation 15 are:

$$\text{Solution 1} \quad n_z = 1; \quad g_\chi = -\frac{B^2}{2\mu_0}\chi_n.$$

Here $\hat{\mathbf{n}}$ is parallel to **B** while the **m** director is free to point in any direction perpendicular to **B**.

$$\text{Solution 2} \quad m_z = 1; \quad g_\chi = -\frac{B^2}{2\mu_0}\chi_m.$$

Here $\hat{\mathbf{m}}$ is parallel to **B** while the **n** director is free to point in any direction perpendicular to **B**.

$$\text{Solution 3} \quad l_z = 1; \quad g_\chi = -\frac{B^2}{2\mu_0}\chi_l.$$

Here $\hat{\mathbf{l}}$ is parallel to **B** while the **n** and the **m** directors are free to point in any compatible direction perpendicular to **B**.

Which of the three solutions is stable follows by determining which minimizes the free energy density. Thus we find that the axis corresponding to the largest magnetic susceptibility points in the direction of the field while the system adopts the same magnetic energy for each possible direction of the other two axes, now perpendicular to the field, and thus the direction of these two axes is not determined by the field.

4. Classification of biaxial nematics with respect to their electromagnetic properties

The discussion of the last section demonstrates that the application of a magnetic or an electric field across a biaxial nematic system does not order the system completely. The aim of this paper is to show how such an ordering can be achieved by the simultaneous application of one magnetic and one electric field. The response of the system in this case, however, depends strongly on how the three dielectric permittivities and the three magnetic susceptibilities are mutually related to each other. Before proceeding it is necessary, therefore, to introduce a classification of the biaxial nematic compounds with respect to their electromagnetic properties.

In order to describe the biaxial nematics we introduce three dielectric permittivities ε_i and three magnetic susceptibilities χ_i. The constants in each of these two groups can be mutually related in six different ways with respect to their magnitudes. Thus there are altogether thirty-six different ways of arranging the six constants ε_i and χ_i. In order to keep track of this ordering we introduce a state vector ψ^{ijk}_{pqr} in the following way. The three upper indices of ψ refer to the ordering of the magnitudes of the dielectric permittivities, while the three lower indices refer to the ordering of the magnitudes of the magnetic susceptibilities. Thus the state vector ψ^{ijk}_{pqr} is used to describe a compound for which $\varepsilon_i > \varepsilon_j > \varepsilon_k$ and $\chi_p > \chi_q > \chi_r$. We now introduce a classification of the thirty-six state vectors in such a way that a state vector is said to belong to one of the three different types depending upon how the ordering of the indices i, j and k is related to the ordering of p, q and r.

Type A

The twenty-four state vectors for which the axis with the largest dielectric permittivity does not coincide with the axis with the largest magnetic susceptiblity, i.e. $i \neq p$. The state vector ψ^{nml}_{mln} (i.e. $\varepsilon_n > \varepsilon_m > \varepsilon_l$, $\chi_m > \chi_l > \chi_n$) is an example of a type A biaxial nematic.

Type B

The six state vectors for which the ordering of the three dielectric permittivities is the same as that of the three magnetic susceptibilities, i.e. $i=p, j=q$, and $k=r$. The state vector ψ_{nml}^{nml}(i.e. $\varepsilon_n>\varepsilon_m>\varepsilon_l$, $\chi_n>\chi_m>\chi_l$) is an example of a type B biaxial nematic.

Type C

The six state vectors for which the axis with the largest dielectric permittivity coincides with the axis with the largest magnetic susceptibility but for which the ordering of the two remaining dielectric permittivities is opposite to that of the magnetic susceptibilities, i.e. $i=p, j=r$ and $k=q$. The state vector ψ_{nlm}^{nml} (i.e. $\varepsilon_n>\varepsilon_m>\varepsilon_l$, $\chi_n>\chi_l>\chi_m$) is an example of a type C biaxial nematic.

The reason for this classification becomes obvious in section 7, where we show that the three different types of biaxial nematics each behave in a unique way when subject to simultaneous electric and magnetic fields.

5. Equilibrium configurations in the presence of two perpendicular electric and magnetic fields

We now derive the possible equilibrium configurations for the system when subject to one electric and one magnetic field, applied to the sample at right angles to each other. In order to be able to decide which field strengths will correspond to the limiting cases strong electric field and strong magnetic field it is useful to introduce a notation that incorporates the coupling constants of the fields into the dielectric permittivities and the magnetic susceptibilities

$$\left.\begin{aligned}\tilde{\varepsilon}_i=\varepsilon_0E^2\varepsilon_i, \qquad \tilde{\varepsilon}_{ij}=\varepsilon_0E^2\varepsilon_{ij}\\ \text{and}\qquad\qquad\\ \tilde{\chi}_i=\mu_0^{-1}B^2\chi_i, \quad \tilde{\chi}_{ij}=\mu_0^{-1}B^2\chi_{ij}.\end{aligned}\right\} \tag{17}$$

Assuming the fields to be applied in the x and z directions, i.e. $\mathbf{E}=E\hat{\mathbf{x}}$ and $\mathbf{B}=B\hat{\mathbf{z}}$, the electromagnetic torques acting on the system calculated from equations (3), (8), (11) and (12) are

$$\left.\begin{aligned}\Gamma_x^{\chi\varepsilon}&=\tilde{\chi}_{nl}n_yn_z+\tilde{\chi}_{ml}m_ym_z,\\ \Gamma_y^{\chi\varepsilon}&=-\tilde{\chi}_{nl}n_xn_z-\tilde{\chi}_{ml}m_xm_z+\tilde{\varepsilon}_{nl}n_xn_z+\tilde{\varepsilon}_{ml}m_xm_z\\ &\text{and}\\ \Gamma_z^{\chi\varepsilon}&=-\tilde{\varepsilon}_{nl}n_xn_y-\tilde{\varepsilon}_{ml}m_xm_y.\end{aligned}\right\} \tag{18}$$

The equilibrium configurations of the system are those for which the torque in equations (18) vanishes. From equations (18) we see immediately that all configurations for which two of the three vectors $\hat{\mathbf{n}}$, $\hat{\mathbf{m}}$ and $\hat{\mathbf{l}}$ are parallel to the fields are possible equilibrium configurations of the system. For certain restricted ranges of the susceptibilities and permittivities, other solutions of these equations may exist, but for the present we prefer to exclude such exceptional cases. It is helpful to introduce a notation for the six solutions in the following way. The solution S_{ij} corresponds to that for which the i director points in the direction of the electric field and the j director is in the direction of the magnetic field. Thus the solution for which $n_x=1$ and $m_z=1$ is denoted S_{nm} and so on. Figure 3 shows the six possible solutions and also gives the appropriate conditions for each of these to be stable. These conditions are derived in the next section.

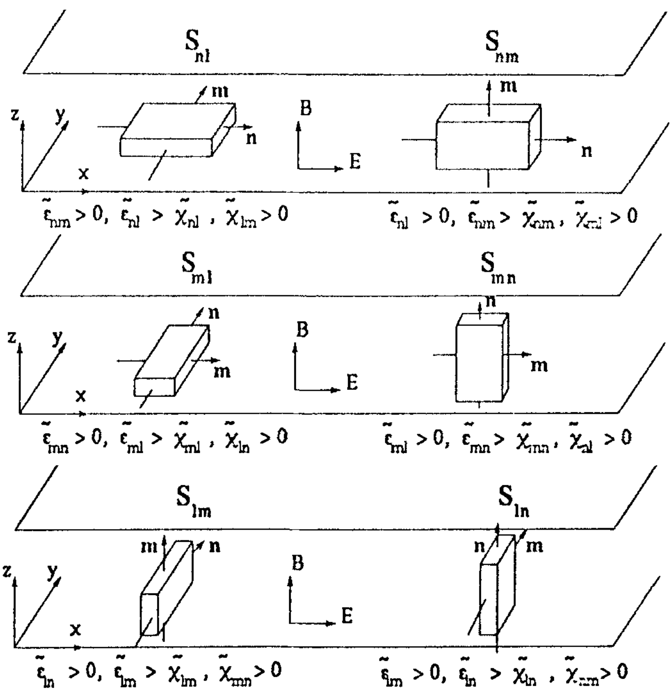

Figure 3. The six equilibrium configurations of biaxial nematics in the presence of two perpendicular electric and magnetic fields. The conditions for each of the solutions to be stable are also displayed in the figure.

6. Stability analysis

In order to investigate the stability of the solutions given by figure 3 we have to study the form of the general torque equation when the system performs small perturbations around these solutions. The question is whether the electromagnetic torque thus developed tends to bring the system back towards equilibrium or not. The starting point of this investigation is the expression of the rotational torque $\mathbf{\Gamma}^r$, i.e. the torque acting on the system when the directors are rotating. The i component of this torque can be written as [7]

$$\Gamma_i^r = \varepsilon_{ijk} t_{kj}^r, \tag{19}$$

where t_{kj}^r is the rotational part of the viscous stress tensor

$$t_{kj}^r = \alpha_2 \dot{n}_k n_j + \alpha_3 \dot{n}_j n_k + \beta_2 \dot{m}_k m_j + \beta_3 \dot{m}_j m_k + \dot{n}_p m_p (\mu_1 m_k n_j + \mu_2 n_k m_j) \tag{20}$$

and α_2, α_3, β_2, β_3, μ_1 and μ_2 are some of the biaxial nematic viscosity coefficients. From equations (19) and (20) we can derive the rotational torque as

$$\left.\begin{aligned}
\Gamma_x^r &= \alpha_2(\dot{n}_z n_y - \dot{n}_y n_z) + \alpha_3(\dot{n}_y n_z - \dot{n}_z n_y) + \beta_2(\dot{m}_z m_y - \dot{m}_y m_z) + \beta_3(\dot{m}_y m_z - \dot{m}_z m_y) \\
&\quad + (\dot{n}_x m_x + \dot{n}_y m_y + \dot{n}_z m_z)[\mu_1(m_z n_y - m_y n_z) + \mu_2(m_y n_z - m_z n_y)], \\
\Gamma_y^r &= \alpha_2(\dot{n}_x n_z - \dot{n}_z n_x) + \alpha_3(\dot{n}_z n_x - \dot{n}_x n_z) + \beta_2(\dot{m}_x m_z - \dot{m}_z m_x) + \beta_3(\dot{m}_z m_x - \dot{m}_x m_z) \\
&\quad + (\dot{n}_x m_x + \dot{n}_y m_y + \dot{n}_z m_z)[\mu_1(m_x n_z - m_z n_x) + \mu_2(m_z n_x - m_x n_z)] \\
\text{and}& \\
\Gamma_z^r &= \alpha_2(\dot{n}_y n_x - \dot{n}_x n_y) + \alpha_3(\dot{n}_x n_y - \dot{n}_y n_x) + \beta_2(\dot{m}_y m_x - \dot{m}_x m_y) + \beta_3(\dot{m}_x m_y - \dot{m}_y m_x) \\
&\quad + (\dot{n}_x m_x + \dot{n}_y m_y + \dot{n}_z m_z)[\mu_1(m_y n_x - m_x n_y) + \mu_2(m_x n_y - m_y n_x)]
\end{aligned}\right\} \tag{21}$$

The relevant equation to study when discussing the stability is the balance of torque equation

$$\mathbf{\Gamma}^r + \mathbf{\Gamma}^{\chi\varepsilon} = 0. \tag{22}$$

A small perturbation of any of the six equilibrium configurations can be expressed by three infinitesimal rotations α, β and γ around the x, y and z axes, respectively. Let us study the solution S_{nl} characterized by $\hat{\mathbf{n}} = \hat{\mathbf{x}}$, $\hat{\mathbf{m}} = \hat{\mathbf{y}}$ and $\hat{\mathbf{l}} = \hat{\mathbf{z}}$. Assuming α, β and γ to be small, the perturbed directors are

$$\left.\begin{aligned} n_x &\approx 1, \\ n_y &\approx \gamma, \\ n_z &\approx -\beta \end{aligned}\right\} \tag{23}$$

and

$$\left.\begin{aligned} m_x &\approx -\gamma, \\ m_y &\approx 1, \\ m_z &\approx \alpha. \end{aligned}\right\} \tag{24}$$

The torque equation (22) in this case reduces via equations (18) and (21) to

$$\left.\begin{aligned} \gamma_m \dot{\alpha} &= -\tilde{\chi}_{lm}\alpha, \\ \gamma_n \dot{\beta} &= -(\tilde{\varepsilon}_{nl} - \tilde{\chi}_{nl})\beta \\ \text{and} \quad \gamma_{nm}\dot{\gamma} &= -\tilde{\varepsilon}_{nm}\gamma, \end{aligned}\right\} \tag{25}$$

where we have introduced the three rotational viscosities [7] of the system which are given by

$$\left.\begin{aligned} \gamma_n &= \alpha_3 - \alpha_2 > 0, \\ \gamma_m &= \beta_3 - \beta_2 > 0 \\ \text{and} \quad \gamma_{nm} &= \alpha_3 - \alpha_2 + \beta_3 - \beta_2 + \mu_2 - \mu_1 > 0. \end{aligned}\right\} \tag{26}$$

By thermodynamical arguments [7] the rotational viscosities must all be positive definite. From equations (25) we see that the perturbation of the system will relax back to the equilibrium $\alpha = \beta = \gamma = 0$ if the coefficients multiplying the right hand sides of these equations are all negative. The stability conditions for the solution S_{nl} therefore read

$$\left.\begin{aligned} \tilde{\varepsilon}_{nm} &> 0, \\ \tilde{\varepsilon}_{nl} &> \tilde{\chi}_{nl} \\ \text{and} \quad \tilde{\chi}_{lm} &> 0. \end{aligned}\right\} \tag{27}$$

By repeating this type of analysis we can derive stability criteria for the other five equilibrium configurations S_{ij}. The results of such an analysis are given in figure 3. We can of course also derive the remaining five stability criteria by the proper interchange of the indices n, m and l in equations (27). The next section discusses the interpretation of the stability conditions given in figure 3 since this is not entirely straightforward.

7. Interpretation of the stability conditions—evidence of bistability

Figure 3 displays the conditions which must be fulfilled if a given equilibrium solution S_{ij} is a stable one. For each solution to be stable there are three conditions. The first and the last of these conditions which are of the type $\tilde{\varepsilon}_{ij}>0$ and $\tilde{\chi}_{ij}>0$ are equivalent to writing $\varepsilon_{ij}>0$ and $\chi_{ij}>0$, and are independent of the field strengths. The middle conditions, which read $\tilde{\varepsilon}_{ij}>\tilde{\chi}_{ij}$ can also be written $\varepsilon_0 E^2\varepsilon_{ij}>\mu_0^{-1}B^2\chi_{ij}$ and are dependent on the ratio between the electric and magnetic field strengths. This makes the analysis of the stability criteria a little more involved and we now show that, depending on the ordering of the dielectric permittivities and the magnetic susceptibilities, a biaxial nematic compound exhibits one out of three completely different qualitative behaviours in the presence of the fields. For the sake of convenience we observe from electromagnetic theory that the relation

$$\varepsilon_0\mu_0=\frac{1}{c^2} \tag{28}$$

is valid, c being the speed of light in a vacuum. In order to measure the ratio between the electric and magnetic field strengths it is helpful to introduce a dimensionless parameter δ according to

$$\delta=\frac{1}{c^2}\left(\frac{E}{B}\right)^2. \tag{29}$$

Our task is now to derive the rules determining which type a given state vector belongs to and which solution is stable for a given δ.

Type A

The state vectors ψ^{ijk}_{pqr} for which $i\neq p$ belong to this type. Compounds belonging to this type always adopt the solution S_{ip} irrespective of the value of δ. This is because there is no conflict between the electric and magnetic torques. As the largest dielectric permittivity belongs to a different axis than the largest magnetic susceptibility, the biaxial plate simply orients itself with the axis with the largest dielectric permittivity parallel to the electric field and the axis with the largest magnetic susceptibility parallel to the magnetic field. There are altogether twenty-four state vectors belonging to this type, and these can be divided into six subgroups, the four state vectors within each subgroup exhibiting the same stable solution.

As an example it is easy to verify that the state vector ψ^{nml}_{lmn} fulfils the stability conditions of the solution S_{nl} but violates the stability conditions of all the other five solutions. Table 1 gives all twenty-four state vectors of type A and their corresponding stable solutions.

Type B

The state vectors with the same ordering for both the dielectric permittivities and the magnetic susceptibilities, i.e. those which can be written ψ^{ijk}_{ijk} belong to this type. Each of these can exhibit one of two stable solutions depending on the ratio δ between the field strengths. The solution S_{ij} is stable for strong electric fields, which in this case means $\delta>\chi_{ij}/\varepsilon_{ij}$. If the magnetic field dominates $(\delta<\chi_{ij}/\varepsilon_{ij})$ the solution S_{ji} is the stable one. This behaviour is also easy to understand. As the largest dielectric permittivity belongs to the same axis as the largest magnetic susceptibility there will be a competition between the electric and magnetic torques. Both fields act to orient the i axis in its direction and the strongest field eventually dominates. The other field then orients the axis with the second largest permittivity or susceptibility.

Table 1. The twenty-four type A state vectors of biaxial nematics and the corresponding stable equilibria. Irrespective of the values of the field strengths, the type A systems exhibit only one stable orientation.

Type A biaxial nematic				
State vector				Stable solution
ψ^{ijk}_{pqr} where $i \neq p$				S_{ip}
ψ^{nml}_{lmn}	ψ^{nml}_{lnm}	ψ^{nlm}_{lmn}	ψ^{nlm}_{lnm}	S_{nl}
ψ^{nlm}_{mln}	ψ^{nlm}_{mnl}	ψ^{nml}_{mln}	ψ^{nml}_{mnl}	S_{nm}
ψ^{mnl}_{lnm}	ψ^{mnl}_{lmn}	ψ^{mln}_{lnm}	ψ^{mln}_{lmn}	S_{ml}
ψ^{mln}_{nlm}	ψ^{mln}_{nml}	ψ^{mnl}_{nlm}	ψ^{mnl}_{nml}	S_{mn}
ψ^{lnm}_{mnl}	ψ^{lnm}_{mln}	ψ^{lmn}_{mnl}	ψ^{lmn}_{mln}	S_{lm}
ψ^{lmn}_{nml}	ψ^{lmn}_{nlm}	ψ^{lnm}_{nml}	ψ^{lnm}_{nlm}	S_{ln}

Table 2. The six type B state vectors of biaxial nematics and the corresponding stable equilibria as they depend on δ. Depending upon whether the electric or the magnetic field is the stronger, the system exhibits one out of two stable solutions.

Type B biaxial nematic		
State vector	Stable solution	
ψ^{ijk}_{ijk}	S_{ij} if $\delta > \frac{\chi_{ij}}{\varepsilon_{ij}}$	S_{ji} if $\delta > \frac{\chi_{ij}}{\varepsilon_{ij}}$
ψ^{nml}_{nml}	S_{nm} if $\delta > \frac{\chi_{nm}}{\varepsilon_{nm}}$	S_{mn} if $\delta < \frac{\chi_{nm}}{\varepsilon_{nm}}$
ψ^{mnl}_{mnl}	S_{mn} if $\delta > \frac{\chi_{mn}}{\varepsilon_{mn}}$	S_{nm} if $\delta < \frac{\chi_{mn}}{\varepsilon_{mn}}$
ψ^{lnm}_{lnm}	S_{ln} if $\delta > \frac{\chi_{ln}}{\varepsilon_{ln}}$	S_{nl} if $\delta < \frac{\chi_{ln}}{\varepsilon_{ln}}$
ψ^{nlm}_{nlm}	S_{nl} if $\delta > \frac{\chi_{nl}}{\varepsilon_{nl}}$	S_{ln} if $\delta < \frac{\chi_{nl}}{\varepsilon_{nl}}$
ψ^{mln}_{mln}	S_{ml} if $\delta > \frac{\chi_{ml}}{\varepsilon_{ml}}$	S_{lm} if $\delta < \frac{\chi_{ml}}{\varepsilon_{ml}}$
ψ^{lmn}_{lmn}	S_{lm} if $\delta > \frac{\chi_{lm}}{\varepsilon_{lm}}$	S_{ml} if $\delta < \frac{\chi_{lm}}{\varepsilon_{lm}}$

An example of a state vector belonging to type B is ψ^{nml}_{nml}. It is easy to convince oneself that this state vector fulfils the stability criteria of the solution S_{nm} if $\delta > \chi_{nm}/\varepsilon_{nm}$, a condition which is equivalent to $\tilde{\varepsilon}_{nm} > \tilde{\chi}_{nm}$. If instead $\delta < \chi_{nm}/\varepsilon_{nm}$ the solution S_{mn} is stable. None of the other four sets of stability conditions can be fulfilled by this state vector. Table 2 gives the six state vectors belonging to type B and also their corresponding stable solutions in terms of their dependence on δ.

Type C

To this type belong the state vectors for which the largest dielectric permittivity and the largest magnetic susceptibility are related to the same axis, but for which the ordering for the two remaining axes are opposite, i.e. the state vectors of the form ψ_{ikj}^{ijk}. Each of these state vectors can exhibit one of two stable solutions depending on the ratio δ between the field strengths. However, the conditions for stability overlap in such a way that both the solutions can be stable simultaneously for some intermediate values of δ. Thus this system exhibits bistability for this range of δ. In this case also, there is a conflicting situation as both fields seek to align the i axis parallel to themselves. It is easy to understand that if the electric field dominates ($\delta \to \infty$) the i axis is parallel to it and the solution S_{ik} is stable. If, on the other hand, the magnetic field dominates ($\delta \to 0$) the i axis is parallel to it and the solution S_{ji} is the stable one. In order to understand the behaviour for the intermediate values of δ where bistability occurs it is helpful to consider figure 4. In this figure the ordering of the parameters, $\varepsilon_i > \varepsilon_j > \varepsilon_k$ and $\chi_i > \chi_k > \chi_j$, is visualized in a level diagram and it is readily verified that the inequalities

$$\frac{\chi_{ik}}{\varepsilon_{ik}} < \frac{\chi_{ik}}{\varepsilon_{ij}} < \frac{\chi_{ij}}{\varepsilon_{ij}} \tag{30}$$

are generally valid for type C systems. The conditions for the solution S_{ik} to be stable can be written

$$\left.\begin{aligned} &\varepsilon_{ij} > 0, \\ &\tilde{\varepsilon}_{ik} > \tilde{\chi}_{ik} \\ \text{and}\qquad & \\ &\chi_{kj} > 0. \end{aligned}\right\} \tag{31}$$

The first and the third of these conditions are trivially fulfilled by the solution S_{ik} as we can see from the level diagram of the figure, and the second condition can be rearranged to read

$$\delta > \frac{\chi_{ik}}{\varepsilon_{ik}}. \tag{32}$$

By permuting the indices in equation (31) we can write the stability conditions for the solution S_{ji} as

$$\left.\begin{aligned} &\varepsilon_{jk} > 0, \\ &\tilde{\varepsilon}_{ji} > \tilde{\chi}_{ji} \\ \text{and}\qquad & \\ &\chi_{ik} > 0. \end{aligned}\right\} \tag{33}$$

Again the first and third conditions are trivially fulfilled by the solution considered and the second of the conditions can be written

$$\delta < \frac{\chi_{ij}}{\varepsilon_{ij}}, \tag{34}$$

where the change of the inequality sign arises from the fact that ε_{ji} is now negative. Thus the solutions S_{ik} and S_{ji} are stable if the conditions from equations (32) and (34) are fulfilled, respectively. From the inequality in equation (30) we see that the lower limit of δ for the solution S_{ik} is always smaller than the upper limit of δ for the solution S_{ji}. Thus, when δ falls between these values the system exhibits bistability.

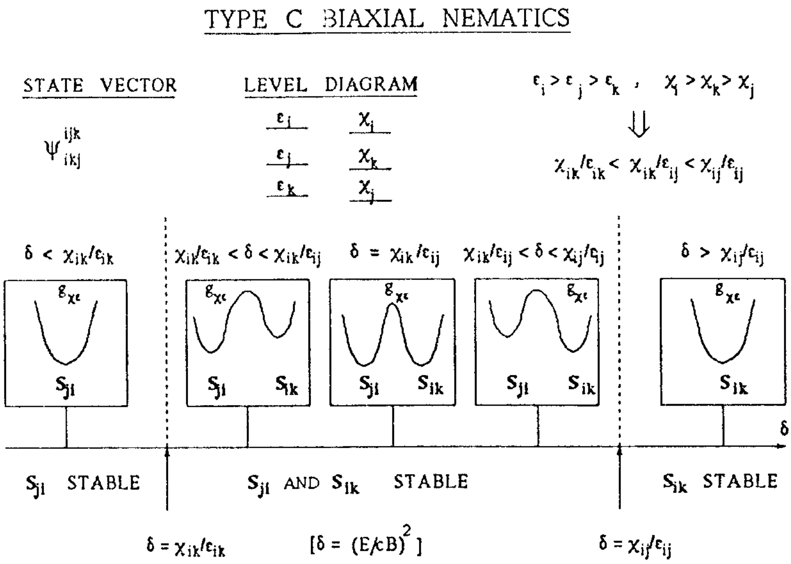

Figure 4. Stability of type C biaxial nematics in the presence of two perpendicular electric and magnetic fields. If the magnetic field dominates ($\delta < \chi_{ik}/\varepsilon_{ik}$), only the solution S_{ji} is stable. If, on the other hand the electric field dominates ($\delta > \chi_{ij}/\varepsilon_{ij}$), only the solution S_{ik} is stable. When the two fields are of comparable strengths ($\chi_{ik}/\varepsilon_{ik} < \delta < \chi_{ij}/\varepsilon_{ij}$), both solutions are stable and the system exhibits bistability. The qualitative behaviour of the electromagnetic free energy density for different values of δ is also shown.

The nature of the bistability is visualized in the energy diagrams in figure 4. These diagrams are only to be interpreted qualitatively as they give one dimensional plots for a three dimensional system. The electromagnetic energy $g_{\chi\varepsilon}$ for the two solutions can be calculated from equations (5) and (9) and is shown to be

$$\left.\begin{aligned} g_{\chi\varepsilon}(S_{ik}) &= -\tfrac{1}{2}(\tilde{\varepsilon}_i + \tilde{\chi}_j) \\ \text{and} \qquad & \\ g_{\chi\varepsilon}(S_{ji}) &= -\tfrac{1}{2}(\tilde{\varepsilon}_j + \tilde{\chi}_i). \end{aligned}\right\} \tag{35}$$

These two energies are equal provided

$$\tilde{\varepsilon}_i + \tilde{\chi}_k = \tilde{\varepsilon}_j + \tilde{\chi}_i, \tag{36}$$

an equality which can be rearranged to read

$$\delta = \frac{\chi_{ik}}{\varepsilon_{ij}}. \tag{37}$$

From the inequality (30) we are assured that this value of δ falls within the bistable region. Thus, when $\delta = \chi_{ik}/\varepsilon_{ij}$ both the solutions are equally stable. When δ decreases the energy of the solution S_{ji} is lower than that of S_{ik} and the latter is, strictly speaking, metastable. Whether a transition from S_{ik} to S_{ji} takes place in this case depends upon the value of the energy maximum between the solutions. However, it is quite obvious that from a practical point of view the system exhibits bistability in an interval around the value $\delta = \chi_{ik}/\varepsilon_{ij}$, the width of which depends on factors not considered in this work.

Table 3. The six type C state vectors of biaxial nematics and the corresponding stable equilibria as they depend on δ. For intermediate values of δ two stable solutions exist and the system exhibits bistability.

Type C biaxial nematic			
State vector	Stable solution		
ψ^{ijk}_{ikj}	S_{ji} if $\delta<\dfrac{\chi_{ik}}{\varepsilon_{ik}}$	S_{ik} and S_{ji} if $\dfrac{\chi_{ik}}{\varepsilon_{ik}}<\delta<\dfrac{\chi_{ij}}{\varepsilon_{ij}}$	S_{ik} if $\delta>\dfrac{\chi_{ij}}{\varepsilon_{ij}}$
ψ^{nml}_{nlm}	S_{mn} if $\delta<\dfrac{\chi_{nl}}{\varepsilon_{nl}}$	S_{nl} and S_{mn} if $\dfrac{\chi_{nl}}{\varepsilon_{nl}}<\delta<\dfrac{\chi_{nm}}{\varepsilon_{nm}}$	S_{nl} if $\delta>\dfrac{\chi_{nm}}{\varepsilon_{nm}}$
ψ^{lnm}_{lmn}	S_{nl} if $\delta<\dfrac{\chi_{lm}}{\varepsilon_{lm}}$	S_{lm} and S_{nl} if $\dfrac{\chi_{lm}}{\varepsilon_{lm}}<\delta<\dfrac{\chi_{ln}}{\varepsilon_{ln}}$	S_{lm} if $\delta>\dfrac{\chi_{ln}}{\varepsilon_{ln}}$
ψ^{mnl}_{mln}	S_{nm} if $\delta<\dfrac{\chi_{ml}}{\varepsilon_{ml}}$	S_{ml} and S_{nm} if $\dfrac{\chi_{ml}}{\varepsilon_{ml}}<\delta<\dfrac{\chi_{mn}}{\varepsilon_{mn}}$	S_{ml} if $\delta>\dfrac{\chi_{mn}}{\varepsilon_{mn}}$
ψ^{nlm}_{nml}	S_{ln} if $\delta<\dfrac{\chi_{nm}}{\varepsilon_{nm}}$	S_{nm} and S_{ln} if $\dfrac{\chi_{nm}}{\varepsilon_{nm}}<\delta<\dfrac{\chi_{nl}}{\varepsilon_{nl}}$	S_{nm} if $\delta>\dfrac{\chi_{nl}}{\varepsilon_{nl}}$
ψ^{lmn}_{lnm}	S_{ml} if $\delta<\dfrac{\chi_{ln}}{\varepsilon_{ln}}$	S_{ln} and S_{ml} if $\dfrac{\chi_{ln}}{\varepsilon_{ln}}<\delta<\dfrac{\chi_{lm}}{\varepsilon_{lm}}$	S_{ln} if $\delta>\dfrac{\chi_{lm}}{\varepsilon_{lm}}$
ψ^{mln}_{mnl}	S_{lm} if $\delta<\dfrac{\chi_{mn}}{\varepsilon_{mn}}$	S_{mn} and S_{lm} if $\dfrac{\chi_{mn}}{\varepsilon_{mn}}<\delta<\dfrac{\chi_{ml}}{\varepsilon_{ml}}$	S_{mn} if $\delta>\dfrac{\chi_{ml}}{\varepsilon_{ml}}$

As an example of a type C state vector consider ψ^{nml}_{nlm}. It is easy to convince oneself that this state vector exhibits the stable solution S_{mn} if $\delta<\chi_{nm}/\varepsilon_{nm}$. If on the other hand $\delta>\chi_{nl}/\varepsilon_{nl}$ the solution S_{nl} is stable. In the overlapping region $\chi_{nl}/\varepsilon_{nl}<\delta<\chi_{nm}/\varepsilon_{nm}$ both solutions are stable, exhibiting an exact bistability in the case $\delta=\chi_{nl}/\varepsilon_{nm}$. None of the other four solutions can be stable for this state vector. Table 3 gives the six state vectors belonging to type C and also the corresponding stable solutions in terms of their dependence upon δ.

8. Discussion

The recently discovered thermotropic biaxial nematic phase provides a fascinating system, the behaviour of which in many respects is similar to that of uniaxial nematics. However, due to the fact that the presence of the transverse director increases the dimensionality of the director field, many phenomena occurring in biaxial nematics will be more complex [7]. Thus we have shown that a biaxial nematic can be assigned one of thirty-six different state vectors in respect of its electromagnetic properties, this is in marked contrast to the four different ways ($\varepsilon_a<0$ or $\varepsilon_a>0$, $\chi_a<0$ or $\chi_a>0$) that are found for the dielectric and magnetic anisotropies of the uniaxial nematic system.

In section 7 the thirty-six state vectors are shown to belong to one of three types, each of which exhibits a qualitatively different behaviour when the system is subject to perpendicular electric and magnetic fields. In the case when the largest dielectric permittivity does not belong to the same axis as the largest magnetic susceptibility, the behaviour of the system is simple and easy to understand (type A). If, however, the largest dielectric permittivity and the largest magnetic susceptibility belong to the same axis, there is a conflict between the electric and magnetic torques and the behaviour is not so easily understood intuitively. However, following an analysis of the stability conditions for the six possible configurations of the system, the type B behaviour is readily accepted. Here the biaxial plate reorients between two stable solutions depending upon whether the electric or magnetic field is dominating as discussed in the previous section. The type C behaviour, however, causes a surprise because here the transition between the stable solutions in the limiting cases of strong electric or magnetic fields is connected with an interval of δ for which both solutions are stable.

One can reach an intuitive understanding of the origin of the difference between type B and type C behaviour in the following way. The two competing solutions for the type B systems are of the form S_{ij} and S_{ji} and these two solutions can be obtained from each other by a simple rotation around the y axis (cf figure 3). The condition for the transition to occur is $\tilde{\varepsilon}_{ij}=\tilde{\chi}_{ij}$. By examining the electromagnetic torque from equation (18) we see that this condition implies $\Gamma_y^{\chi\varepsilon}=0$, and thus at the transition all orientations for which the i and j axes are confined within the plane defined by the fields actually correspond to zero torque. In the language of phase transitions this corresponds to the transition being continuous or of second order. For the type C system on the other hand the situation is more complicated. The two competing solutions are of the form S_{ik} and S_{ji} and these two solutions cannot be obtained from each other by a simple rotation around any of the symmetry axes of the system. Examining the electromagnetic torque from equation (18) we see that for the two values of δ for which one of the solutions ceases to be stable, there is no route towards the other stable equilibrium for

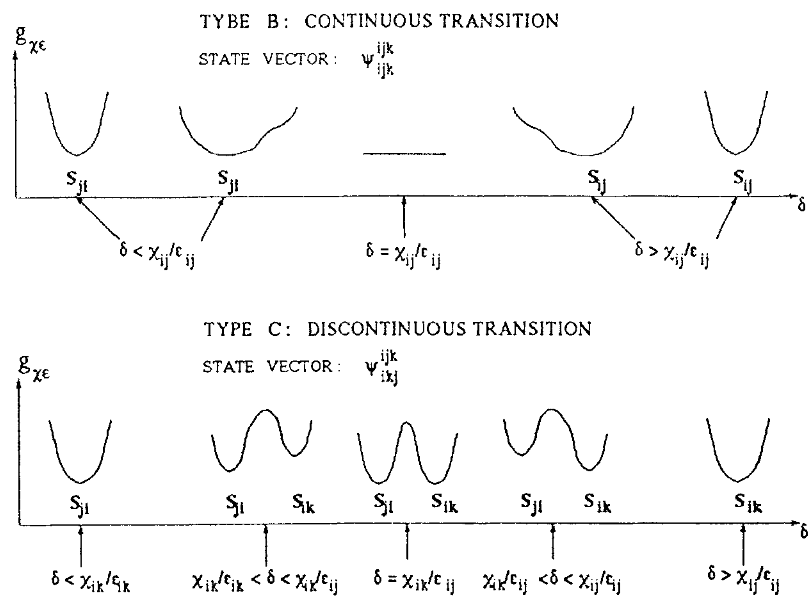

Figure 5. The qualitative behaviour of the electromagnetic potential plotted for different values of δ. This illustrates the difference in behaviour for the transition between stable states for type B and type C biaxial nematics. The transition for type B systems is continuous and the exchange of stability takes place for one given value of δ. For type C systems there is an intermediate interval of δ for which two solutions are stable and the transition between them is discontinuous. The value of δ for which the transition occurs depends upon whether δ is increasing or decreasing, and thus the transition exhibits a hysteresis effect.

which the torque is overall zero as was the case for the type B behaviour. Thus the situation corresponds to a first order or discontinuous transition. Such transitions are known to exhibit the feature of coexistence of more than one stable state. In figure 5 we have sketched sequences of electromagnetic potential diagrams for type B and C behaviour in order to visualize the difference between the two cases.

One important lesson to be learned from this paper is that when trying to orient a biaxial nematic sample by the means of electric and magnetic fields, difficulties can arise if sufficient care is not exercised. For example, if attempting to control the directors by the use of electric and magnetic fields in a flow experiment, it is necessary to use strong fields to overcome the viscous torque. Dealing with a type A biaxial nematic causes no problems, because there is always one unique, stable orientation irrespective of the field strengths for this system. For the type B biaxial nematic some problems could arise, because while it is necessary to use strong fields, we probably also require a value of δ for which $\delta \gg \chi_{ij}/\varepsilon_{ij}$ or $\delta \ll \chi_{ij}/\varepsilon_{ij}$. This is because if $\delta \approx \chi_{ij}/\varepsilon_{ij}$ for a type B system, the electromagnetic orientational torque around the y axis is small due to the competition between the fields. We can therefore find ourselves in a situation where it is necessary to compromise, because if δ is too close to $\chi_{ij}/\varepsilon_{ij}$ we have a poor orientational effect from the fields, although still wishing to keep both fields as strong as possible to have the electromagnetic torque large compared to the viscous one. For the type C biaxial nematic the situation is even more complex. Again it may be necessary to compromise concerning the field strengths, but even worse, in the bistable region of δ, the orientation obtained depends upon the configuration of the system prior to the application of the fields. Thus we must carefully examine the situation in this case and apply the fields in the proper order to achieve the orientation desired. When bistability occurs, initial application of an electric field orients the i axis parallel to the field. By then applying the magnetic field, the other two axes of the biaxial plate are ordered. In this way we obtain the solution S_{ik}. By reversing the order of application of the fields, the solution S_{ji} is obtained instead.

Finally, it is worth mentioning that there are many problems associated with electromagnetic torques in biaxial nematics that we have not addressed in this paper. One interesting problem to investigate are the shapes of the attracting domains in the three dimensional director space in the case of bistability. Such an investigation would be an additional guide to the experimentalist who wants to orient a biaxial nematic sample. Other interesting problems to study are the torque patterns and the distribution and stability of equilibrium orientations in the case of competing electromagnetic and viscous torques. Our opinion is that the behaviour of the system in the latter case will exhibit many interesting features, and show a much more complex behaviour than is the case for the uniaxial nematic phase.

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